Available online at http://cna-journal.com
Commun. Nonlinear Anal. 2 (2020), 120–130
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.cna-journal.com

Location of Zeros of Polynomials

P. Ramulu[a,*], G.L.Reddy[b]

[a] *Department of Mathematics, M.V.S Govt. Arts and Science College(Autonomous), Mahabubnagar, Telangana, India-509001.*
[b] *School of Mathematics and Statistics, University of Hyderabad, India-500046.*

Abstract

In this paper we consider the problem of finding the estimation of maximum number of zeros in a prescribed region and the results which we obtain generalize and improve upon some well known results.

Keywords: Zeros of polynomial, Eneström- Kakeya theorem, Prescribed region.
2010 MSC: 30C10, 30C15

1. Introduction

According to the Eneström- Kakeya [2,5]: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0<a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$ then all the zeros of P(z) lie in $|z| \leq 1$ and concerning the number of zeros of the polynomial in the region $|z| \leq \frac{1}{2}$, the following result is due to Mohammad [6].

Theorem 1.1. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0<a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$. Then the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, does not exceed*

$$1 + \frac{1}{log2} log \frac{a_n}{a_0}.$$

In this paper we want to prove the following results. In fact the following results generalize some of the results in [3,7-12].

2. Main results

Theorem 2.1. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2},\ i = 0, 1, 2, ..., n,\ for\ some\ real\ \beta,\ a_0 \neq 0\ and$$

*Corresponding author
Email addresses: `ramulu.purra@gmail.com` (P. Ramulu 1), `glrsm@uohyd.ernet.in` (G.L.Reddy)

Received 2020-02-11

$$|a_n| \geq |a_{n-1}| \leq |a_{n-2}| \geq |a_{n-3}| \leq ... \geq |a_{n-m+1}| \leq |a_{n-m}| \geq |a_{n-m-1}| \geq ... \geq |a_2| \geq |a_1| \geq |a_0|$$

if both n and (n-m) are even or odd, (OR)

$$|a_n| \geq |a_{n-1}| \leq |a_{n-2}| \geq |a_{n-3}| \leq ... \leq |a_{n-m+1}| \geq |a_{n-m}| \geq |a_{n-m-1}| \geq ... \geq |a_2| \geq |a_1| \geq |a_0|$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.
Then (i) the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$, *does not exceed*

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_1 cos\alpha}{|a_0|}$$

if both n and (n-m) are even or odd

$$where\ Y_1 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)],$$

OR

(ii) the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$, *does not exceed*

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_2 cos\alpha}{|a_0|}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

$$where\ Y_2 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m}| + |a_{n-m+1}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+4}| + |a_{n-m+2}|)].$$

Corollary 2.2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq m \geq 2$ *with complex coefficients such that* $|arg(a_i)| \leq \frac{\pi}{2}$ $for\ i = 1, 2, .., n, a_0 \neq 0$ *and*

$$|a_n| \geq |a_{n-1}| \leq |a_{n-2}| \geq ... \geq |a_{n-m+1}| \leq |a_{n-m}| \geq |a_{n-m-1}| \geq ... \geq |a_2| \geq |a_1| \geq |a_0|$$

if both n and (n-m) are even or odd, (OR)

$$|a_n| \geq |a_{n-1}| \leq |a_{n-2}| \geq ... \leq |a_{n-m+1}| \geq |a_{n-m}| \geq |a_{n-m-1}| \geq ... \geq |a_2| \geq |a_1| \geq |a_0|$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.
Then (i) the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, *does not exceed*

$$1 + \frac{1}{log2} log \frac{|a_n| + Y_1}{|a_0|}\ if\ both\ n\ and\ (n-m)\ are\ even\ or\ odd,$$

$$where\ Y_1 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)],$$

OR

(ii) the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, *does not exceed*

$$1 + \frac{1}{log2} log \frac{|a_n| + Y_2}{|a_0|},$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even,

$$where\ Y_2 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m}| + |a_{n-m+1}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+4}| + |a_{n-m+2}|)].$$

Remark 2.3. By taking $\alpha = \beta = 0$ and $r_1 = \frac{1}{2}$ in Theorem 2.1, it reduces to Corollary 2.2

Theorem 2.4. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2},\ i = 0, 1, 2, ..., n,\ for\ some\ real\ \beta,\ a_0 \neq 0\ and$$

$$|a_n| \leq |a_{n-1}| \geq |a_{n-2}| \leq ... \leq |a_{n-m+1}| \geq |a_{n-m}| \geq |a_{n-m-1}| \geq ... \geq |a_2| \geq |a_1| \geq |a_0|$$

if both n and (n-m) are even or odd,

OR

$$|a_n| \leq |a_{n-1}| \geq |a_{n-2}| \leq ... \geq |a_{n-m+1}| \leq |a_{n-m}| \geq |a_{n-m-1}| \geq ... \geq |a_2| \geq |a_1| \geq |a_0|$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.
Then (i) the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$, does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_3 cos\alpha}{|a_0|}$$

if both n and (n-m) are even or odd,

$$where\ Y_3 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_n| + |a_{n-2}| + ... + |a_{n-m+4}| + |a_{n-m+2}|)].$$

OR

(ii) the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$, does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_4 cos\alpha}{|a_0|}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even,

$$\begin{aligned} where\ Y_4 =& [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+2}| + |a_{n-m}|) \\ & - (|a_n| + |a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)]. \end{aligned}$$

Corollary 2.5. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with complex coefficients such that $|arg(a_i)| \leq \frac{\pi}{2}$, for $i = 0, 1, 2, ..., n$, $a_0 \neq 0$ and*

$$|a_n| \leq |a_{n-1}| \geq |a_{n-2}| \leq |a_{n-3}| \geq ... \leq |a_{n-m+1}| \geq |a_{n-m}| \geq |a_{n-m-1}| \geq ... \geq |a_2| \geq |a_1| \geq |a_0|$$

if both n and (n-m) are even or odd, (OR)

$$|a_n| \leq |a_{n-1}| \geq |a_{n-2}| \leq |a_{n-3}| \geq ... \geq |a_{n-m+1}| \leq |a_{n-m}| \geq |a_{n-m-1}| \geq ... \geq |a_2| \geq |a_1| \geq |a_0|$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.
Then (i) the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, does not exceed

$$1 + \frac{1}{log2} log \frac{|a_n| + Y_3}{|a_0|}\ if\ both\ n\ and\ (n-m)\ are\ even\ or\ odd,$$

$$where\ Y_3 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_n| + |a_{n-2}| + ... + |a_{n-m+4}| + |a_{n-m+2}|)],$$

OR

(ii) the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, *does not exceed*

$$1 + \frac{1}{log2} log \frac{|a_n| + Y_4}{|a_0|},$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

$$\begin{aligned} where\ Y_4 =& [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+2}| + |a_{n-m}|) \\ & - (|a_n| + |a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)]. \end{aligned}$$

Remark 2.6. By taking $\alpha = \beta = 0$ and $r_1 = \frac{1}{2}$ in Theorem 2.4, it reduces to Corollary 2.5.

Theorem 2.7. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq m \geq 2$ *with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2},\ i = 0, 1, 2, ..., n,\ for\ some\ real\ \beta,\ a_0 \neq 0\ and$$

$$|a_n| \geq |a_{n-1}| \leq |a_{n-2}| \geq |a_{n-3}| \leq ... \geq |a_{n-m+1}| \leq |a_{n-m}| \leq |a_{n-m-1}| \leq ... \leq |a_2| \leq |a_1| \leq |a_0|$$

if both n and (n-m) are even or odd (OR)

$$|a_n| \geq |a_{n-1}| \leq |a_{n-2}| \geq |a_{n-3}| \leq ... \leq |a_{n-m+1}| \geq |a_{n-m}| \leq |a_{n-m-1}| \leq ... \leq |a_2| \leq |a_1| \leq |a_0|$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.
Then (i) the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$, *does not exceed*

$$\frac{1}{log\frac{1}{r_1}} log \frac{(|a_n| + |a_0|)(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n-1} |a_i| + 2\lambda_1 cos\alpha}{|a_0|}$$

if both n and (n-m) are even or odd

$$where\ \lambda_1 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)],$$

OR

(ii) the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$, *does not exceed*

$$\frac{1}{log\frac{1}{r_1}} log \frac{(|a_n| + |a_0|)(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n-1} |a_i| + 2\lambda_2 cos\alpha}{|a_0|}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

$$where\ \lambda_2 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+2}| + |a_{n-m}|)].$$

Corollary 2.8. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq m \geq 2$ *with complex coefficients such that* $|arg(a_i)| \leq \frac{\pi}{2},\ for\ i = 0, 1, 2, ..., n,\ a_0 \neq 0\ and$

$$|a_n| \geq |a_{n-1}| \leq |a_{n-2}| \geq ... \geq |a_{n-m+1}| \leq |a_{n-m}| \leq |a_{n-m-1}| \leq ... \leq |a_2| \leq |a_1| \leq |a_0|$$

if both n and (n-m) are even or odd,

OR

$$|a_n| \geq |a_{n-1}| \leq |a_{n-2}| \geq ... \leq |a_{n-m+1}| \geq |a_{n-m}| \leq |a_{n-m-1}| \leq ... \leq |a_2| \leq |a_1| \leq |a_0|$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.
Then (i) the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, *does not exceed*

$$1 + \frac{1}{log2} log \frac{|a_n| + |a_0| + \lambda_1}{|a_0|}\ if\ both\ n\ and\ (n-m)\ are\ even\ or\ odd,$$

$$where\ \lambda_1 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)],$$

OR

(ii) the number of zeros of P(z) in $|z| \leq \frac{1}{2}$*, does not exceed*

$$1 + \frac{1}{log2} log \frac{|a_n| + |a_0| + \lambda_2}{|a_0|},$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even,

$$where\ \lambda_2 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+2}| + |a_{n-m}|)].$$

Remark 2.9. By taking $\alpha = \beta = 0$ and $r_1 = \frac{1}{2}$ in Theorem 2.7, it reduces to Corollary 2.8.

Theorem 2.10. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq m \geq 2$ *with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2},\ i = 0, 1, 2, ..., n,\ for\ some\ real\ \beta,\ a_0 \neq 0\ and$$

$$|a_n| \leq |a_{n-1}| \geq |a_{n-2}| \leq |a_{n-3}| \geq ... \leq |a_{n-m+1}| \geq |a_{n-m}| \leq |a_{n-m-1}| \leq ... \leq |a_2| \leq |a_1| \leq |a_0|$$

if both n and (n-m) are even or odd,

OR

$$|a_n| \leq |a_{n-1}| \geq |a_{n-2}| \leq |a_{n-3}| \geq ... \geq |a_{n-m+1}| \leq |a_{n-m}| \leq |a_{n-m-1}| \leq ... \leq |a_2| \leq |a_1| \leq |a_0|$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.
Then (i) the number of zeros of P(z) in $|z| \leq r_1$*,* $0 < r_1 < 1$*, does not exceed*

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n} |a_i| + 2\lambda_3 cos\alpha}{|a_0|},$$

if both n and (n-m) are even or odd,

$$where\ \lambda_3 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|)],$$

OR

(ii) the number of zeros of P(z) in $|z| \leq r_1$*,* $0 < r_1 < 1$*, does not exceed*

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n} |a_i| + 2\lambda_4 cos\alpha}{|a_0|}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even,

$$where\ \lambda_4 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+4}| + |a_{n-m+2}|) - (|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)].$$

Corollary 2.11. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq m \geq 2$ *with complex coefficients such that* $|arg(a_i)| \leq \frac{\pi}{2},\ for\ i = 0, 1, 2, ..., n,\ a_0 \neq 0$ *and*

$$|a_n| \leq |a_{n-1}| \geq |a_{n-2}| \leq |a_{n-3}| \geq ... \leq |a_{n-m+1}| \geq |a_{n-m}| \leq |a_{n-m-1}| \leq ... \leq |a_2| \leq |a_1| \leq |a_0|$$

if both n and (n-m) are even or odd, (OR)

$$|a_n| \leq |a_{n-1}| \geq |a_{n-2}| \leq |a_{n-3}| \geq ... \geq |a_{n-m+1}| \leq |a_{n-m}| \leq |a_{n-m-1}| \leq ... \leq |a_2| \leq |a_1| \leq |a_0|$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.
Then (i) the number of zeros of P(z) in $|z| \leq \frac{1}{2}$*, does not exceed*

$$1 + \frac{1}{log2} log \frac{|a_0| + \lambda_3}{|a_0|}\ if\ both\ n\ and\ (n - m)\ are\ even\ or\ odd,$$

where $\lambda_3 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|)]$,

(OR) (ii) the number of zeros of P(z) in $|z| \leq \frac{1}{2}$, *does not exceed*

$$1 + \frac{1}{log2} log \frac{|a_0| + \lambda_4}{|a_0|},$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even,

where $\lambda_4 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+4}| + |a_{n-m+2}|) - (|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)]$.

Remark 2.12. By taking $\alpha = \beta = 0$ and $r_1 = \frac{1}{2}$ in Theorem 2.10, it reduces to Corollary 2.11.

We need the following lemmas for proof of the Theorems.

3. Lemmas

Lemma 3.1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients such that*

$$|arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2};\ |a_{i-1}| \leq |a_i|\ for\ some\ i = 0, 1, 2, ..., n.$$

$$Then\ |a_i - a_{i-1}| \leq (|a_i| - |a_{i-1}|)cos\alpha + (|a_i| + |a_{i-1}|)sin\alpha.$$

The above lemma is due to Govil [4].

Lemma 3.2. [1] : *If* $f(z)$ *is regular* $f(0) \neq 0$ *and* $|f(z)| \leq M$ $(M>0)$ *in* $|z| \leq 1$ *then the number of zeros of* $f(z)$ *in* $|z| \leq r_1$, $0<r_1<1$ *does not exceed*

$$\frac{1}{log\frac{1}{r_1}} log \frac{M}{|f(0)|}.$$

4. Proof of the Theorems

Proof of Theorem 2.1

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ *be a polynomial of degree* $n \geq 2$.

Then consider the polynomial $Q(z) = (1 - z)P(z)$ *so that*

$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0.$

Then for $|z| > 1$, *we have*

$$
\begin{aligned}
|Q(z)| &\leq |a_n| + \sum_{i=1}^{n} |a_i - a_{i-1}| + |a_0| \\
&= |a_n| + |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + ... + |a_{n-m+1} - a_{n-m}| + \sum_{i=1}^{n-m} |a_i - a_{i-1}| \\
&\leq |a_n| + (|a_n| - |a_{n-1}|)cos\alpha + (|a_n| + |a_{n-1}|)sin\alpha + (|a_{n-2}| - |a_{n-1}|)cos\alpha \\
&\quad + (|a_{n-2}| + |a_{n-1}|)sin\alpha + (|a_{n-2}| - |a_{n-3}|)cos\alpha + (|a_{n-2}| + |a_{n-3}|)sin\alpha \\
&\quad + ... + (|a_{n-m+2}| - |a_{n-m+1}|)cos\alpha + (|a_{n-m+2}| + |a_{n-m+1}|)sin\alpha \\
&\quad + (|a_{n-m}| - |a_{n-m+1}|)cos\alpha + (|a_{n-m}| + |a_{n-m+1}|)sin\alpha + ... + \sum_{i=1}^{n-m} (|a_i| - |a_{i-1}|)cos\alpha \\
&\quad + \sum_{i=1}^{n-m} (|a_i| + |a_{i-1}|)sin\alpha + |a_0| \ \textit{if both n and } (n-m) \textit{ are even or odd (by Lemma 3.1)} \\
&= |a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_1 cos\alpha - |a_0|(cos\alpha + sin\alpha - 1) \\
&\leq |a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_1 cos\alpha.
\end{aligned}
$$

where $Y_1 = (|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|)$
$- (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)].$

By applying Lemma 3.2 to $Q(z)$, we get that the number of zeros of $Q(z)$ in $|z| \leq r_1$, $0 < r_1 < 1$ does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_1 cos\alpha}{|a_0|}$$

if both n and (n-m) are even or odd

where $Y_1 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)].$

Since the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$ is equal to the number of zeros of $Q(z)$ in $|z| \leq r_1$, if both n and (n-m) are even or odd, we get the required result.

Similarly we can prove that if n is even and (n-m) is odd (or) if n is odd and (n-m) is even, by re-arranging the terms in the above proof.

That is the number of zeros of $Q(z)$ in $|z| \leq r_1$, $0 < r_1 < 1$ does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_2 cos\alpha}{|a_0|}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even,

where $Y_2 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+4}| + |a_{n-m+2}|)].$

This completes the proof of Theorem 2.1

Proof of Theorem 2.4

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ *be a polynomial of degree* $n \geq 2$.

Then consider the polynomial $Q(z) = (1-z)P(z)$ *so that*

$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0.$

Then for $|z|>1$, *we have*

$$\begin{aligned}
|Q(z)| &\leq |a_n| + \sum_{i=1}^{n} |a_i - a_{i-1}| + |a_0| \\
&= |a_n| + |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + ... + |a_{n-m+1} - a_{n-m}| \\
&\quad + \sum_{i=1}^{n-m} |a_i - a_{i-1}| \\
&\leq |a_n| + (|a_{n-1}| - |a_n|)cos\alpha + (|a_{n-1}| + |a_n|)sin\alpha + (|a_{n-1}| - |a_{n-2}|)cos\alpha \\
&\quad + (|a_{n-1}| + |a_{n-2}|)sin\alpha + (|a_{n-3}| - |a_{n-2}|)cos\alpha + (|a_{n-3}| + |a_{n-2}|)sin\alpha \\
&\quad + ... + (|a_{n-m+1}| - |a_{n-m+2}|)cos\alpha + (|a_{n-m+1}| + |a_{n-m+2}|)sin\alpha \\
&\quad + (|a_{n-m+1}| - |a_{n-m}|)cos\alpha + (|a_{n-m+1}| + |a_{n-m}|)sin\alpha + ... + \sum_{i=1}^{n-m} (|a_i| - |a_{i-1}|)cos\alpha \\
&\quad + \sum_{i=1}^{n-m} (|a_i| + |a_{i-1}|)sin\alpha + |a_0|
\end{aligned}$$

if both n and $(n-m)$ *are even or odd (by Lemma* 3.1 *)*

$$\begin{aligned}
&= |a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_3 cos\alpha - |a_0|(cos\alpha + sin\alpha - 1) \\
&\leq |a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_3 cos\alpha,
\end{aligned}$$

$$\text{where } Y_3 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_n| + |a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+4}| + |a_{n-m+2}|)]$$

By applying Lemma 3.2 to $Q(z)$, we get that the number of zeros of $Q(z)$ in $|z| \leq r_1$, $0 < r_1 < 1$ does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_3 cos\alpha}{|a_0|}.$$

if both n and (n-m) are even or odd, where

$$Y_3 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_n| + |a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+4}| + |a_{n-m+2}|)]$$

Since the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$ is equal to the number of zeros of $Q(z)$ in $|z| \leq r_1$, if both n and (n-m) are even or odd, we get the required result.

Similarly we can prove that if n is even and (n-m) is odd (or) if n is odd and (n-m) is even, by re-arranging the terms in the above proof.

That is the number of zeros of $Q(z)$ in $|z| \leq r_1$, $0 < r_1 < 1$ does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_n|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=0}^{n-1} |a_i| + 2Y_4 cos\alpha}{|a_0|}.$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even, where

$$Y_4 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+2}| + |a_{n-m}|) - (|a_n| + |a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)]$$

.

This completes the proof of Theorem 2.4.

Proof of Theorem 2.7

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ *be a polynomial of degree* $n \geq 2$.
Then consider the polynomial $Q(z) = (1 - z)P(z)$ *so that*
$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0$.
Then for $|z|>1$, *we have*

$$|Q(z)| \leq |a_n| + \sum_{i=1}^{n} |a_i - a_{i-1}| + |a_0|$$

$$= |a_n| + |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + ... + |a_{n-m+1} - a_{n-m}| + \sum_{i=1}^{n-m} |a_i - a_{i-1}|$$

$$\leq |a_n| + (|a_n| - |a_{n-1}|)cos\alpha + (|a_n| + |a_{n-1}|)sin\alpha + (|a_{n-2}| - |a_{n-1}|)cos\alpha + (|a_{n-2}| + |a_{n-1}|)sin\alpha + (|a_{n-2}| - |a_{n-3}|)cos\alpha + (|a_{n-2}| + |a_{n-3}|)sin\alpha + ... + (|a_{n-m+2}| - |a_{n-m+1}|)cos\alpha + (|a_{n-m+2}| + |a_{n-m+1}|)sin\alpha + (|a_{n-m}| - |a_{n-m+1}|)cos\alpha + (|a_{n-m}| + |a_{n-m+1}|)sin\alpha + ... + \sum_{i=1}^{n-m} (|a_{i-1}| - |a_i|)cos\alpha + \sum_{i=1}^{n-m} (|a_{i-1}| + |a_i|)sin\alpha + |a_0|$$

if both n *and* $(n - m)$ *are even or odd (by Lemma* 3.1)

$$= (|a_n| + |a_0|)(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n-1} |a_i| + 2\lambda_1 cos\alpha$$

where $\lambda_1 = [(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)]$

By applying Lemma 3.2 to $Q(z)$, we get that the number of zeros of $Q(z)$ in $|z| \leq r_1$, $0 < r_1 < 1$ does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{(|a_n| + |a_0|)(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n-1} |a_i| + 2\lambda_1 cos\alpha}{|a_0|}.$$

if both n and (n-m) are even or odd, where

$$\lambda_1 = \left[(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+2}| + |a_{n-m}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)\right].$$

Since the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$ is equal to the number of zeros of $Q(z)$ in $|z| \leq r_1$, if both n and (n-m) are even or odd, we get the required result.

Similarly we can prove that if n is even and (n-m) is odd (or) if n is odd and (n-m) is even, by re-arranging the terms in the above proof.

That is the number of zeros of $Q(z)$ in $|z| \leq r_1$, $0 < r_1 < 1$ does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{(|a_n| + |a_0|)(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n-1} |a_i| + 2\lambda_2 cos\alpha}{|a_0|}.$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even, where

$$\lambda_2 = \big[(|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+2}| + |a_{n-m}|)\big].$$

This completes the proof of Theorem 2.7.

Proof of Theorem 2.10

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ *be a polynomial of degree* $n \geq 2$.
Then consider the polynomial $Q(z) = (1 - z)P(z)$ *so that*

$$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0.$$

Then for $|z|>1$, *wehave*

$$\begin{aligned}
|Q(z)| \leq\ & |a_n| + \sum_{i=1}^{n} |a_i - a_{i-1}| + |a_0| \\
=\ & |a_n| + |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + ... + |a_{n-m+1} - a_{n-m}| \\
& + \sum_{i=1}^{n-m} |a_i - a_{i-1}| \\
\leq\ & |a_n| + (|a_{n-1}| - |a_n|)cos\alpha + (|a_{n-1}| + |a_n|)sin\alpha + (|a_{n-1}| - |a_{n-2}|)cos\alpha \\
& + (|a_{n-1}| + |a_{n-2}|)sin\alpha + (|a_{n-3}| - |a_{n-2}|)cos\alpha + (|a_{n-3}| + |a_{n-2}|)sin\alpha \\
& + ... + (|a_{n-m+1}| - |a_{n-m+2}|)cos\alpha + (|a_{n-m+1}| + |a_{n-m+2}|)sin\alpha \\
& + (|a_{n-m+1}| - |a_{n-m}|)cos\alpha + (|a_{n-m+1}| + |a_{n-m}|)sin\alpha \\
& + ... + \sum_{i=1}^{n-m} (|a_{i-1}| - |a_i|)cos\alpha + \sum_{i=1}^{n-m} (|a_{i-1}| + |a_i|)sin\alpha + |a_0| \\
& \textit{if both n and } (n-m) \textit{ are even or odd (by Lemma } 3.1) \\
=\ & |a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n} |a_i| + 2\lambda_3 cos\alpha - |a_n|(cos\alpha + sin\alpha - 1) \\
\leq\ & |a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n} |a_i| + 2\lambda_3 cos\alpha
\end{aligned}$$

where $\lambda_3 = [(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+4}| + |a_{n-m+2}| + |a_{n-m}|)]$.

By applying Lemma 3.2 to $Q(z)$, we get that the number of zeros of $Q(z)$ in $|z| \leq r_1$, $0 < r_1 < 1$ does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n} |a_i| + 2\lambda_3 cos\alpha}{|a_0|}.$$

if both n and (n-m) are even or odd, where

$$\lambda_3 = \big[(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+3}| + |a_{n-m+1}|) - (|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+4}| + |a_{n-m+2}| + |a_{n-m}|)\big].$$

Since the number of zeros of P(z) in $|z| \leq r_1$, $0 < r_1 < 1$ is equal to the number of zeros of $Q(z)$ in $|z| \leq r_1$, if both n and (n-m) are even or odd, we get the required result.

Similarly we can prove that if n is even and (n-m) is odd (or) if n is odd and (n-m) is even, by re-arranging the terms in the above proof.

That is the number of zeros of $Q(z)$ in $|z| \le r_1$, $0 < r_1 < 1$ does not exceed

$$\frac{1}{log\frac{1}{r_1}} log \frac{|a_0|(cos\alpha + sin\alpha + 1) + 2sin\alpha \sum_{i=1}^{n} |a_i| + 2\lambda_4 cos\alpha}{|a_0|}.$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even, where

$$\lambda_4 = \left[(|a_{n-1}| + |a_{n-3}| + ... + |a_{n-m+4}| + |a_{n-m+2}|) - (|a_{n-2}| + |a_{n-4}| + ... + |a_{n-m+3}| + |a_{n-m+1}|)\right].$$

This completes the proof of Theorem 2.10.

References

[1] K. K. Dewan, Extremal Properties and Coefficient Estimates for Polynomials with Restricted Zeros and on Location of Zeros of Polynomials, Ph.D Thesis, Indian Institutes of Technology, Delhi, 1980

[2] G.Eneström,Remarquee sur un théorème relatif aux racines de l'equation $a_n + ... + a_0 = 0$ oü tous les coefficient sont et positifs, Tôhoku Math.J 18 (1920), 34-36.

[3] G.L.Reddy ,P.Ramulu, On The Zeros of Polynomials,Int Jr. of Mathematical Sciences and Applications, 5(2), (2015), 395-399.

[4] N.K.Govil and Q.I.Rehman, On the EnstromKakeya Theorem Tohoku Math .J. 20, (1968), 126-136.

[5] S.Kakeya, On the limits of the roots of an alegebraic equation with positive coefficient, Tôhoku Math.J, (2), (1912-1913), 140-142.

[6] Q.G.Mohammad, On the Zeros of the Polynomials,American Mathematical Monthly, 72(6), (1965), 631-633.

[7] P.Ramulu, G.L. Reddy, and C.Gangadhar, Location of Zeros of Analytic Functions, American Journal of Engineering Research, 4(7), (2015), 83-87.

[8] P.Ramulu, G.L. Reddy, and C.Gangadhar, On the Zeros of Polynomials with Real Coefficients, International Journal of Applied Mathematical Sciences, 5(2),(2015), 109-118.

[9] P.Ramulu, G.L. Reddy, and C.Gangadhar, Zeros of Polynomials with Restricted Coefficients, IOSR Journal of Mathematics, 11(2), Ver. V, (2015), 21-28 .

[10] P.Ramulu, G.L.Reddy, Location of zeros of Polynomials with with Restricted Real Coefficients,International J. of Math. Sci. and Engg. Appls., 9(II), (2015), 1-12.

[11] P.Ramulu, G.L.Reddy , On The Enestrom-Kakeya Theorem, International Journal of Pure and Applied Mathematics, 102(4), (2015), 687-700.

[12] P.Ramulu, Some Generalization of Eneström-Kakeya Theorem, International Journal of Mathematics and Statistics Invention, 3(2),(2015), 52-59.